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"One of three parts, but they are unequal": Elementary School Teachers' Understanding of Unit Fractions*†

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ABSTRACT

Teachers' mathematical knowledge is critical for their teaching of mathematics. We investigated whether elementary teachers believe that a unit fraction, $1/n$, results only from a whole equipartitioned into n parts. We adapted Ciosek and Samborska's (2016) Frame Task, presenting a frame consisting of three unequal segmented squared rings, with one squared ring shaded. In semi-structured interviews, 19 teachers engaged the task and reasoned whether the shaded portion equals 1/3 of the frame. Our findings indicate that about three-quarters of the participants believe that either (1) to have one-third of a quantity, a section needs to be one of three parts, or (2) a section cannot be 1/3 of an object if the object is partitioned into three unequal sections. Finally, we hypothesize how an iterative perspective of unit fractions from a measuring perspective may mitigate against the false beliefs that Ciosek and Samborska (2016) and our study document.

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*We dedicate this work to the memory of our beloved friend and colleague Muteb M. Alqahtani (1982-2022), whose untimely death occurred after we completed this report.

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"Uma das três partes, mas são desiguais": Compressão dos Professores dos Anos Iniciais do Ensino Fundamental sobre Frações Unitárias

RESUMO

O conhecimento matemático dos professores é fundamental para o ensino da Matemática. Investigamos se os professores dos anos iniciais do Ensino Fundamental acreditam que uma fração unitária, $1/n$, resulta apenas de um todo equipartado em n partes. Adaptamos o *Frame Task* de Ciosek e Samborska (2016), apresentando um quadro composto por três anéis quadrados segmentados desigualmente, com um anel quadrado sombreado. Em entrevistas semiestruturadas, 19 professores se engajaram na tarefa e raciocinaram se a parte sombreada é igual a 1/3 do quadro. Nossos resultados indicam que cerca de três quartos dos participantes acreditam que (1) para ter um terço de uma quantidade, uma seção precisa ser uma das três partes ou (2) uma seção não pode ser 1/3 de um objeto se o objeto for particionado em três seções desiguais. Finalmente, levantamos a hipótese de como uma perspectiva iterativa de frações unitárias a partir de uma perspectiva de medição pode mitigar as falsas crenças que Ciosek e Samborska (2016) e nosso estudo documentam.

Palavras-chave: Conceitos e Operações Numéricas; Conhecimento Docente; Crenças do Professor; Números racionais; Perspectiva de Medição.

"Una de tres partes, pero son desiguales": La comprensión de las fracciones unitaria por los maestros de escuela primaria

RESUMEN

El conocimiento matemático de los maestros es fundamental para su enseñanza de las Matemáticas. Investigamos si los maestros de primaria creen que una fracción unitaria, $1/n$, resulta solo de un todo equiparticionado en n partes. Adaptamos Frame Task de Ciosek y Samborska (2016), presentando un marco compuesto por tres anillos cuadrados segmentados desiguales, con un anillo cuadrado sombreado. En entrevistas semiestructuradas, 19 docentes se involucraron en la tarea y razonaron si la parte sombreada equivale a $1/3$ del marco. Nuestros resultados indican que alrededor de las tres cuartas partes de los participantes creen que (1) para ser un tercio de una cantidad, una sección debe ser una de tres partes, o (2) una sección no puede ser $1/3$ de un objeto si el objeto se divide en tres secciones desiguales. Finalmente, planteamos la hipótesis de cómo una perspectiva iterativa de fracciones unitarias desde una perspectiva de medición puede mitigar las falsas creencias que documentan Ciosek y Samborska (2016) y nuestro estudio.

Palabras clave: Conceptos y operaciones numéricas; Conocimiento del maestro; Creencias de los maestros; Números racionales; Perspectiva de medición

INTRODUCTION

Foundational interpretations of what a fraction is are numerous and interwoven. They are pedagogical, historical, mathematical, and neurocognitive (Cuisenaire & Gattegno, 1954; Davydov & Tsvetkovich, 1991; Kieren, 1976, 1980; Matthews & Chesney, 2015). Those interpretations shape how learners perceive a whole or unit and an associated unit fraction; that is, a fraction with 1 as its numerator. Studies illustrate how assigning a unit of measure to a given quantity, or unitizing, is critical for working adeptly with fractions (Lamon, 1996, 2007; Mack, 2001; Olive, 1999; Steffe & Olive, 2010; Van Ness & Alston, 2017a, 2017b, 2017c). To inaugurate students' study of fractions, textbooks and disciplinary policy (Common Core State Standards Initiative, 2010) present the part/whole interpretation. However, research suggests that a fraction's part/whole understanding leads learners to conceive and represent with difficulty fractions greater than a unit or whole (Gabriel et al., 2012, 2013; Mack, 1990; Tzur, 1999; Zhang et al., 2017). To understand further the difficulty of learners to conceive and represent improper fractions, a potentially fruitful line of inquiry concerns investigating how a part/whole interpretation of fractions shapes learners' comprehension of a unit fraction. Exploring an aspect of unit fraction comprehension, Ciosek and Samborska (2016) present a hitherto undocumented belief among elementary to university students, science graduates, and mathematics teachers that a unit fraction, $1/n$, results only from a whole equipartitioned into n parts. Using Ciosek and Samborska's (2016) study instrument, our work extends theirs by examining the related ideas of practicing elementary school teachers. Teachers of those grades are responsible for supporting elementary school students' development of ideas about fractions and their operations. Therefore, understanding teachers' ideas may provide valuable insights into the origins of students' beliefs and indicate percipient ways to challenge and enhance students' fundamental awareness of unit fractions.

RELATED LITERATURE AND CONCEPTUAL FRAMEWORK

Teachers' Mathematical Knowledge

Teachers' mathematical knowledge is critical for teaching mathematics and students' achievement. Several models have theorized the knowledge that teachers mobilize to teach mathematics effectively. Our study draws from the Mathematics Teacher's Specialised Knowledge (MTSK) model by Carrillo-Yañez et al. (2018). Extending the work of Shulman (1986) and Ball et al. (2008), Carrillo-Yañez et al. (2018) conceptualize teachers' knowledge as consisting of three major domains: beliefs, pedagogical content knowledge, and mathematical knowledge. Since teachers' beliefs influence teaching practice, MTSK centers the two other domains around teachers' beliefs about mathematics and mathematics teaching and learning. Each of the remaining two domains includes three subdomains. The pedagogical content knowledge domain involves knowledge of features of learning mathematics (KFLM), knowledge of mathematics teaching (KMT), and the knowledge of mathematics learning standards (KMLS). The KFLM subdomain encompasses the features inherent to learning certain mathematical content. The second subdomain, KMT, concerns awareness of mathematical teaching theories, including knowing about activities, strategies, and techniques for teaching specific mathematical content. The last subdomain, KMLS, is the knowledge about mathematical standards and curricula for the content at different levels.

The mathematical knowledge domain includes knowledge of topics (KoT), knowledge of the structure of mathematics (KSM), and knowledge of practices in mathematics (KPM). The KSM subdomain describes teachers' knowledge about the relations among different mathematical ideas. This knowledge influences teachers' decisions when connecting a mathematical topic to previous or future topics. Carrillo-Yañez et al. (2018) define the KPM subdomain as "any mathematical activity carried out systematically, which represents a pillar of mathematical creation and which conforms to a logical basis from which rules can be extracted" (p. 9). This knowledge does not focus on teaching mathematics but on the workings of mathematics, such as mathematical demonstrations, justifications, definitions, and making deductions and inductions. Finally, the KoT subdomain describes "*what* and *in what way* the mathematics teacher knows [or may know] the topics they teach" (Carrillo-Yañez et al., 2018, p. 7, original emphasis). An example of this type of knowledge includes concepts, procedures, and theorems.

This study investigates the mathematical knowledge domain and focuses on teachers' KoT. The specific topic of our research concerns teachers' knowledge of the unit fraction concept, its relation to a whole and other parts of the whole. Moreover, regarding elementary school teachers who teach mathematics, this study also seeks to explore Ciosek and Samborska's (2016) hypothesis: "An iterative procedure of dissecting something into n equal parts to constitute a fraction of $1/n$ (as it is defined) may lead to the false belief of the learner of mathematics that if a whole is divided into n unequal parts, none of them can be $1/n$ of its size" (p. 22). Investigating how teachers conceptualize and identify unit fractions is essential since, as Chapman (2014) states, "it is not only important what mathematics teachers know but also how they know it and what they are able to mobilize for teaching" (p. 295).

Unit Fractions

Fractions, one of three representations of rational numbers, have several interpretations. Kieren (1976, 1980) calls them sub-constructs and identifies five: part-whole, quotient, measures, ratios, and operators. These interpretations, Kieren further notes, are united by the act of equipartitioning a whole. "Partitioning is seen here as any general strategy for dividing a given quantity into a given number of 'equal' parts. Thus, it can be seen as important in developing all of the five sub-constructs" (1976, p. 138). In that quote and elsewhere in our text, the term *quantity* means a measurable quality of an object such as its length, area, or volume. Positing partitioning a quantity as the cognitive basis for fraction knowledge has, in practice, implied that instructionally the part/whole interpretation is the foundational and initial fraction concept. This stance perseveres as curricular policy (Common Core State Standards Initiative, 2010).

This partitioning approach Vergnaud (1983) identifies as the first of two categories of ratios or fractions as comparisons or proportions. The first category he calls inclusive fractions, represented by a whole and a part of it (p out of q) such as this set model statement: Aaliyah ate two-thirds of the cookies. For an area model, one compares pizza slices to the whole pizza. The part/whole interpretation is the ubiquitous basis for initial learning about fractions among students in the United States and elsewhere. The inclusive category is how fractional parts of a whole are understood, including unit fractions. For example, the unit fraction, $1/n$, quantifies the part-whole relationship. It represents one part of the whole's equal parts. Its quantification or magnitude results from equipartitioning a whole and considering the ratio between one part and the whole.

In contrast, Vergnaud (1983) denotes the second category of fraction representations as exclusive. In it, fractions multiplicatively compare two distinct quantities with no inclusion relationship (p to q)—for example, the volume of Karma's luggage is three-fifths of Samir's luggage. The second quantity is the unit to which the first is compared. The quantities are of the same kind (volume) and compared proportionally. As a further instance, Davydov and Tsvetkovich (1991) present this situation: A student compares two distinct objects, sharing length as a common attribute, a ruler to a table's side. To compare the objects, the student assigns a quantity to equal one and measures, actions corresponding to the fraction concept's origin. For example, if n iterations of the student's ruler equal the table's side, and she considers the table the unit, the ruler is one-nth or $1/n$ of the table.

Building on the preceding paragraph's conceptual ideas, we subscribe to a generalized formulation of a unit fraction. A unit fraction, $1/n$, quantifies a particular multiplicative relation between two quantities of the same kind (e.g., lengths, areas, or collections). Specifically, the multiplicative relation is where n iterations of one quantity measure the other quantity considered the unit of measurement. In other words, definitionally, a quantity is $1/n$ of a unit if and only if n iterations of the quantity equal the unit. Though this definition integrates inclusive and exclusive categories of fractions and their corresponding nonsymbolic models, the crucial difference between the two categories is whether the n iterations are internal or external to the unit quantity. In either case, a unit fraction is a unit of a unit or a subunit.

Research evidence learners' movement from inclusive to exclusive fraction representation. This movement is essential for, as Vergnaud (1983) notes, "comparisons and ratios between any two quantities of the same kind are a more powerful model than inclusive fractions, providing a more general foundation for scalar operators or ratios" (p. 164). Hunt et al. (2016) show how tasks for learners with learning disabilities can conceptually prime them to use unit fractions to construct non-unit fractions in and out of equal sharing contexts. Tzur's (1999) study evinces how children nontrivially reorganize their numerical operations with a unit fraction to construct fractions less than or equal to a whole to represent fractional magnitudes greater than a whole.

Other than initiating fraction learning with the part/whole interpretation and its inclusive representations, challenging the settled partitioning perspective of fraction learning, there are instructional research and pedagogical materials that develop fractional knowledge with exclusive models (Cuisenaire & Gattegno, 1954; Davydov & Tsvetkovich, 1991; Dougherty & Simon, 2014; Dougherty & Venenciano, 2007).

Assigning a quantity to equal one or a unit is a mental and bodily act. Lamon (1996) calls the "assignment of a unit of measurement to a given quantity" (p. 170) unitizing. For a given whole or unit, a fraction whose numerator is 1, a unit fraction, is a subunit. From the part/whole perspective, if a whole is partitioned into n equal parts, one part is $1/n$ of the whole. As Ciosek and Samborska's (2016) study suggests, from that statement, learners may incorrectly conclude the statement's converse: If one part of a whole is $1/n$ of it, then the whole is divided into n equal parts. Besides, learners may falsely believe versions of the following two statements:

- If a whole is partitioned into n parts, one part is $1/n$ of the whole.
- If a whole is partitioned into n unequal parts, one cannot equal $1/n$ of the whole.

The part/whole interpretation of a unit fraction means that if both conditions—n parts and all parts equal—are true for how a unit is partitioned, each part equals $1/n$ of the unit. However, the statement does not imply that a part cannot equal $1/n$ of the unit when the unit is not partitioned into n parts or the parts are unequal. That is, a part of a unit can equal $1/n$ of it even (a) when the unit is partitioned into n unequal parts or (b) without the unit being partitioned into n parts.

To engage learners' awareness of the meaning and conditions for the existence of unit fractions, educators can develop instructional tasks that challenge potential false beliefs (see, for example, Problems 1 to 6 in Ciosek and Samborska (2016, pp. 30-31). However, to support the appropriate implementation of those tasks, it is essential to know how elementary school teachers understand the consequences of a fraction's part/whole interpretation for ideas about unit fractions. Then, if needed, teacher educators can support teachers to be aware of what the interpretation means and does not mean about the existence of unit fractions.

METHODS

Study Instrument

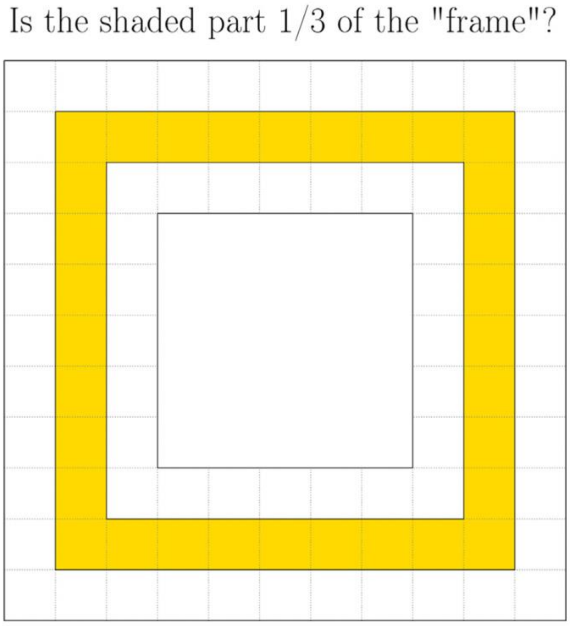
To explore how elementary school teachers understand unit fractions and, if necessary, support them in educating their awareness of false ideas, we implemented the Frame Task (see Figure 1 below), adapted from the study instrument in Ciosek and Samborska (2016, p. 22), which they adopted from a Polish fourth-grade textbook. This task reveals participants' unit fraction understanding as one of n equal parts of a whole. It also allows us to examine whether they have the false belief that if a whole is divided into n unequal parts, none of them will equal $1/n$ of the whole.

In Figure 1, the frame consists of three squared annuli or rings—an outer, a middle or shaded, and an inner—each segmented into squares of the same size. Not part of the frame is the unsegmented center square.

Participants and Setting

From a pool of 47 teachers of grades 1 to 5 participating in a larger professional learning project about rational numbers, this study involved 19 of the teachers. They are from a public charter school in the Bronx, New York City, with two locations, which we call B1 and B2. They volunteered to participate in individual interviews, responding to the Frame Task, justifying and illustrating their responses, reacting to challenges to their answers or justifications, examining alternative explanations, and discussing their takeaways from the interview experience. For the interview, the teachers meet with one of the two first authors for an average of 18.2 minutes during the school day through videoconferencing. All interviews occurred on the same day. So that participating teachers did not receive information in advance about the task and the nature of the interview, we asked each participant not to reveal their experience to their colleagues until the next school day. All participating teachers signed an online informed-consent form previously approved by the Institutional Review Board of Rutgers University.

Figure 1 - The Frame Task

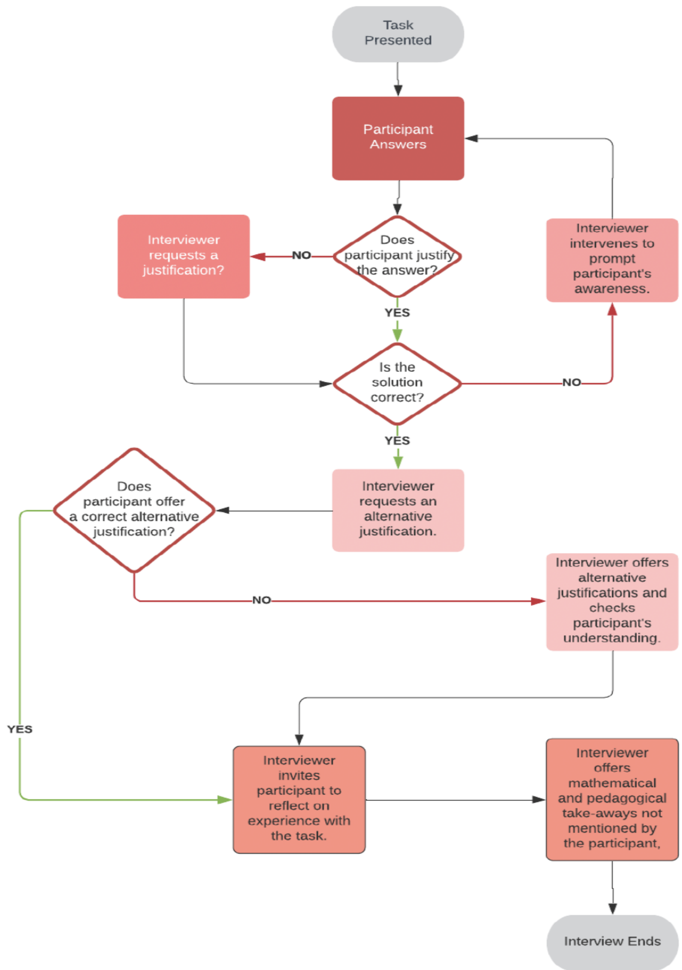


Data Production

Guided by the interview procedure from Ciosek and Samborska (2016), the current study’s two interviewers performed semi-structured interviews (Lune & Berg, 2017). We randomly assigned each participant to one of the two researchers for a one-half-hour time slot. Each researcher interviewed teachers from both B1 and B2, following a protocol displayed in Figure 2. At the appointed time, teachers logged into the videoconferencing application, and their interviewer informed them that the session would be videorecorded. In addition, the interviewers confirmed that teachers knew how to use the application’s annotation tool to illustrate their work and then shared their screens to display the Frame Task.

After displaying the task, we asked participants to read the question—Is the shaded part 1/3 of the “frame”—and, for the context, explain their understanding of the word “frame.” Their explanation was essential since the word as a noun or verb has several meanings. Furthermore, even when considering a picture frame, the consideration is variable as to which portion of the figure constitutes the frame in question. Once the participant and interviewer agreed that, for the task, the "frame" was the area of the figure surrounding the central large white square, the interviewer asked the participant to respond to the task. A complete response entailed two components: an answer to the task's question and a justification of the answer. Therefore, aside from their answers, we listened to how participants justified them. If they did not justify their answer, we then invited them to do so. If the participants' answers and justifications were correct, we encouraged them to provide an alternative rationale for their answers. If they did not have one, we offered an alternative explanation based on counting or mentally decomposing and recomposing the task's graphical figure.

Figure 2 – Interview Protocol Flow Chart



If participants' answers or justifications were incorrect, we prompted their awareness by offering a monetary scenario that challenged their erroneous reasonings, such as whether a unit fraction can only be defined for an equally partitioned object. After participants responded to the challenge and revised their answer or justification, we invited them to provide an alternative rationale for their answer. If they had none, we offered an alternative justification.

Finally, the interview ended with participant and researcher reflections. First, we encouraged participants to reflect on their experience in the interview from the perspectives of mathematics, learning, and teaching. Second, to complement or extend what the participants said, we offered mathematical perspectives to contribute further to their KoT of fractions.

Our mathematical perspectives comprise two considerations. The first KoT idea concerns this statement: If a whole is partitioned into n equal parts, one part is $1/n$ of the whole. This partition or part/whole interpretation of a unit fraction means that if both conditions—n parts and all parts equal—are true for how a unit whole is partitioned, each part equals $1/n$ of the unit. However, the statement does not imply that a part cannot equal $1/n$ of the unit when either the unit is not partitioned into n parts or when the unit's parts are unequal. That is, a part of a unit can equal $1/n$ of it (a) without the unit being partitioned into n parts or (b) even when the unit is partitioned into n unequal parts.

The second KoT consideration we offered pertains to the following generalized interpretation of unit fraction: A quantity is $1/n$ of a unit if and only if n iterations of the quantity equal the unit. We hypothesize that this understanding of a unit fraction, including that it is considered a subunit, emerges naturally from a measuring perspective of fraction knowledge (Powell, 2019b, 2023).

DATA ANALYSIS

The data consisted of the video recordings of the online interviews, including participants' annotations on a digital version of the Frame Task and transcripts of the videos' audio tracks. To probe the interview data, the authors developed an analysis spreadsheet containing columns for components of the interview and a row for each participant. The columns pertinent to this report captured participants' initial answers, whether their justification was unprompted or prompted and its content, whether the interviewer challenged their answer or explanation and the content of the challenge, and the conclusion of how a participant addressed the challenge. In addition, two of the authors independently analyzed the interview data. For their analyses, the inter-rater reliability measure was 93.42% agreement, and the Cohen's Kappa coefficient is 0.858, indicating strong agreement between the two raters (McHugh, 2012).

RESULTS

We report results corresponding to segments of our interview that parallel those Ciosek and Samborska (2016) detailed. Consequently, we do not discuss participants' reflections about alternative explanations for why the frame's yellow-shaded portion is one-third of it that they offered or that we supplied and comments about what they learned from the interview.

A complete response to the Frame Task entails two parts, an answer to the task's question and its justification. However, our data analysis suggests that a small proportion of the participants initially responded correctly to the Frame Task, and most teachers struggled at first to reason correctly about the conditions necessary for a portion of an object to represent a unit fraction.

A participant's initial responses comprised an answer to the question—"Is the shaded part 1/3 of the 'frame?'"—and the reason for their answer. A response essentially was either "yes" or "no," coupled with a justification. Nevertheless, two participants' initial answers were a version of "Let me check" and then responded "yes" followed by a justification. Table 1 contains a tally of participants who provided each of the three initial responses and provides justification examples from teachers, T19, T5, and T3. In Table 1, of the 11 who responded "yes," about 32 percent or six provided incorrect justifications, noted with the letter "I" next to a paraphrasing of their explanation, and the five others or roughly 26 percent proffered correct reasonings, indicated with the letter "C."

As participants' initial responses, the incorrect to correct responses were 26 percent to 74 percent. The eight teachers who initially answered "no" justified their answers, stating that the three squared rings of the frame are not the same size. Among participants whose initial answer was "yes," we observed incorrect and valid justifications. Six teachers argued incorrectly that the frame's shaded part is $1/3$ of the frame since it is one of three parts. In sum, though not uttered by a participant, the composite of the two incorrect justifications is "one of three parts, but they are unequal."

Amongst the correct explanations were those who answered "yes" or "Let me check." Aside from the six participants who answered affirmative but supplied incorrect reasoning, three participants answered “yes” and reasoned correctly by comparing the number of shaded squares (32) to the total number of squares (96).

The two teachers who first answered some version of “Let me check.” Each one compared the number of the shaded squares to the total number of squares in the frame before answering. That is, they inspected the frame, compared the three squared rings, and stated that the shaded part is one-third of the frame. Then, they mentally shifted squares from the frame’s outer ring to its inner ring and verified that the shifting resulted in two squared rings the same size as the shaded squared ring. For example, one of the “Let-me-check” teachers, T12, reasoned by mentally decomposing and recomposing the outer and inner squared rings to visualize that the average number of small squares is the same for the frame’s three squared rings.

Table 1 - Initial responses and examples

Initial Answer	Justification	Count	Example
Yes	(I) The shaded part is one of three parts.	6 (31.6%)	T19: “the shaded area represents one of three parts.”
	(C) The number of small shaded squares equals one-third of the total number of squares.	3 (15.8%)	T5: “So, the yellow part of this frame is 32... So, it is one-third of 96.”
No	The three parts are not the same size.	8 (42.1%)	T3: “I see three divisions of the frame, but they’re not all equal.”
Let me check-Yes	The number of shaded squares to the total number of squares is 1/3.	2 (10.5%)	
Totals		19 (100%)	

Two teachers who first answered some version of “Let me check.” Each one compared the number of the shaded squares to the total number of squares in the frame before answering. That is, they inspected the frame, compared the three squared rings, and stated that the shaded part is one-third of the frame. Then, they mentally shifted squares from the frame's outer ring to its inner ring and verified that the shifting resulted in two squared rings the same size as the shaded squared ring. For example, one of the "Let-me-check" teachers, T12, reasoned by mentally decomposing and recomposing the outer and inner squared rings to visualize that the average number of small squares is the same for the frame's three squared rings.

For participants who responded incorrectly, we prompted their awareness by offering one of two scenarios that challenged their erroneous reasoning: (a) Suppose you divide $90 into these three parts: $45, $30, and $15. Is any of these 1/3 of $90? or (b) Suppose you divide $90 into these three parts: $40, $35, $15. Is any of the parts 1/3 of $90? We gave the first scenario to participants whose initial answer was "no," and offered the second scenario to participants who answered "yes" but incorrectly justified. In all cases, after we challenged their thinking with one of the two scenarios, participants revised either their answer or justification or both and, ultimately, responded correctly based on the portion of the shaded square ring to the total of the three squared rings.

DISCUSSION

A goal of our study was to investigate how elementary teachers who teach mathematics understand how unit fractions are constituted. Specifically, we were interested in knowing whether they believe that a unit fraction, $1/n$, results only from a whole equipartitioned into n parts, as Ciosek and Samborska (2016) documented among a range of elementary to university students, science graduates, and mathematics teachers. Our work adds to the literature as it inquired into a population not included in the Ciosek and Samborska (2016) study, namely elementary school teachers who teach mathematics, those who initiate students' formal study of fractions.

Our inquiry is crucial since teachers' understanding shapes the learning opportunities they offer their students (Borko et al., 1992; Fisher, 1988). Moreover, a domain of teachers' specialized knowledge for teaching mathematics, knowledge of topics (KoT), includes what of and how they know the topics they teach (Carrillo-Yañez et al., 2018). Therefore, understanding elementary school teachers' ideas, KoT of fractions, may provide valuable insights into the origins of students' non-normative beliefs about fractions. Furthermore, discerning whether teachers have robust conceptual insights into the unit fraction concept will enable mathematics teacher educators to know whether pre-service and in-service teachers need support to engage students with counterexamples to challenge mistaken notions about unit fractions.

Our study and Ciosek and Samborska's (2016) results indicate that, respectively, about 75% and 65% of the participants initially harbor incorrect interpretations of how the part/whole perspective defines unit fractions. Specifically, in our study, elementary-grade teachers who teach mathematics believe either (1) for a section of an object to equal one-third of it, the section needs to be one of three parts of the object or (2) a section cannot be equal to one-third of an object if the object is partitioned into three unequal sections. Notably, when we presented scenarios to challenge those conceptual beliefs, the teachers revised their thinking and responded correctly to the task.

Our investigation suggests that, like teachers, without working through counterexamples, students may develop incorrect conclusions about the constitution of unit fractions, such as this idea: The only way to obtain $1/n$ of a given whole is to divide it into n equal parts. Educators will want to challenge and enhance students' fundamental awareness of how unit fractions are constituted.

FINAL CONSIDERATIONS

Earlier in this report, we mentioned the following generalized interpretation of a unit fraction: A quantity is $1/n$ of a unit if and only if n iterations of the quantity equal the unit. From a part/whole perspective, Tzur (1999) analyzes a constructivist teaching experiment to show how children, engaging in an iterative fraction scheme (Olive, 1999), nontrivially reorganize their numerical operations with a unit fraction to construct fractions less than or equal to a whole then later to represent fractional magnitudes greater than a whole. This cognitive feat is noteworthy since the fundamental basis for conceiving the unit fraction is the equipartitioning of a whole. That criterion causes learners to hesitate iterating a unit fraction beyond the magnitude of the whole (Gabriel et al., 2012, 2013; Mack, 1990; Tzur, 1999; Zhang et al., 2017) and, as the present study and Ciosek and Samborska (2016) show, perceiving a unit fraction among parts of a whole partitioned unequally.

To attenuate cognitive obstacles associated with founding fractions on equipartitioning a quantity, we hypothesize that learners' iterative fraction scheme, handling comeasurement units (Olive, 1999), and construction of proper, improper, and mixed numbers emerge naturally from a measuring perspective of fraction knowledge (Alqahtani & Powell, 2018; Powell, 2019a, 2019b, 2023). This perspective defines fractions as ratios that express the multiplicative comparison of two quantities of the same kind. We surmise that working from that perspective, learners spontaneously encounter the meaning of a general fraction, m/n, as an outcome of m iterations of the unit fraction $1/n$. Our research team intends to explore this hypothesis in a future study. The study is part of research program challenging the settled partitioning perspective of fraction learning inherent in inclusive models of fraction representation while ignoring potentially more cognitively powerful exclusive models occasioned by a measuring perspective of fraction knowledge.

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REFERENCES

Alqahtani, M. M., & Powell, A. B. (2018). Investigating how a measurement perspective changes pre-service teachers' interpretations of fractions. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 484 - 487). University of South Carolina & Clemson University.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education, 59*(5), 389-407.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to Teach Hard Mathematics: Do Novice Teachers and Their Instructors Give up Too Easily? *Journal for Research in Mathematics Education, 23*(3), 194-222. https://doi.org/10.2307/749118

Carrillo-Yañez, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, Á., Ribeiro, M., & Muñoz-Catalán, M. C. (2018). The mathematics teacher's specialised knowledge (MTSK) model. *Research in Mathematics Education, 20*(3), 236-253. https://doi.org/10.1080/14794802.2018.1479981

Chapman, O. (2014). Overall Commentary: Understanding and Changing Mathematics Teachers. In J.-J. Lo, K. R. Leatham, & L. R. Van Zoest (Eds.), *Research Trends in Mathematics Teacher Education* (pp. 295-309). Springer. https://doi.org/10.1007/978-3-319-02562-9_16

Ciosek, M., & Samborska, M. (2016). A false belief about fractions – What is its source? *The Journal of Mathematical Behavior*, *42*, 20-32. https://doi.org/https://doi.org/10.1016/j.jmathb.2016.02.001

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics* National Governors Association Center for Best Practices, Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Cuisenaire, G., & Gattegno, C. (1954). *Numbers in colour: A new method of teaching the process of arithmetic to all levels of the Primary School*. Hienemann.

Davydov, V. V., & Tsvetkovich, Z. H. (1991). The object sources of the concept of fractions. In V. V. Davydov & L. P. Steffe (Eds.), *Soviet studies in mathematics education. Volume 6. Psychological abilities of primary school children in learning mathematics* (pp. 86-147). National Council of Teachers of Mathematics.

Dougherty, B., & Simon, M. (2014). Elkonin and Davydov Curriculum in Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 204-207). Springer Netherlands. https://doi.org/10.1007/978-94-007-4978-8_56

Dougherty, B. J., & Venenciano, L. C. H. (2007). Measure up for understanding: Reflect and discuss. *Teaching Children Mathematics*, *13*(9), 452-456. https://doi.org/10.5951/TCM.13.9.0452

Fisher, L. C. (1988). Strategies Used by Secondary Mathematics Teachers to Solve Proportion Problems. *Journal for Research in Mathematics Education*, *19*(2), 157-168. https://doi.org/10.2307/749409

Gabriel, F., Coché, F., Szucs, D., Carette, V., Rey, B., & Content, A. (2012). Developing children’s understanding of fractions: An intervention study. *Mind, Brain, and Education*, *6*(3), 137-146. https://doi.org/10.1111/j.1751-228X.2012.01149.x

Gabriel, F., Coché, F., Szucs, D., Carette, V., Rey, B., & Content, A. (2013). A componential view of children's difficulties in learning fractions. *Frontiers in Psychology*, *4*, 1-12. https://doi.org/10.3389/fpsyg.2013.00715

Hunt, J. H., Tzur, R., & Westenskow, A. (2016). Evolution of Unit Fraction Conceptions in Two Fifth-Graders with a Learning Disability: An Exploratory Study. *Mathematical Thinking and Learning*, *18*(3), 182-208. https://doi.org/10.1080/10986065.2016.1183089

Kieren, T. E. (1976). On the mathematical, cognitive and instructional foundations of rational number. In R. A. Lesh (Ed.), *Number and measurement* (pp. 101-144). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kieren, T. E. (1980). The rational number construct-Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125-150). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27*(2), 170-193. https://doi.org/10.5951/jresematheduc.27.2.0170

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In J. F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A Project of the National Council of Teachers of Mathematics* (pp. 629-667). Information Age.

Lune, H., & Berg, B. L. (2017). *Qualitative research methods for the social sciences* (9th ed.). Pearson.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*(1), 16-32. https://doi.org/10.5951/jresematheduc.21.1.0016

Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education, 32*(3), 267-295. https://doi.org/10.2307/749828

Matthews, P. G., & Chesney, D. L. (2015). Fractions as percepts? Exploring cross-format distance effects for fractional magnitudes. *Cognitive Psychology, 78*, 28-56. https://doi.org/10.1016/j.cogpsych.2015.01.006

McHugh, M. L. (2012). Interrater reliability: the kappa statistic. *Biochemia medica: Biochemia medica, 22*(3), 276-282.

Olive, J. (1999, 1999/12/01). From Fractions to Rational Numbers of Arithmetic: A Reorganization Hypothesis. *Mathematical Thinking and Learning, 1*(4), 279-314. https://doi.org/10.1207/s15327833mtl0104_2

Powell, A. B. (2019a). How does a fraction get its name? *Revista Brasileira de Educação em Ciências e Educação Matemática, 3*(3), 700-713.

Powell, A. B. (2019b). Measuring perspective of fraction knowledge: Integrating historical and neurocognitive findings. *Revista Sergipana de Matemática e Educação Matemática, 4*(1), 1-19.

Powell, A. B. (2023). Two perspectives of fraction knowledge: characterization, origins and implications. *Caminhos da Educação Matemática em Revista, 11*(1), 76-92.

Shulman, L. S. (1986). Those who understand knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Steffe, L. P., & Olive, J. (Eds.). (2010). *Children's fractional knowledge*. Springer.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education*, *30*(4), 390-416.

Van Ness, C. K., & Alston, A. S. (2017a). Establishing the importance of the unit. In C. A. Maher & D. Yankelewitz (Eds.), *Children's reasoning while building fraction ideas* (pp. 49-64).

Van Ness, C. K., & Alston, A. S. (2017b). Justifying the choice of the unit. In C. A. Maher & D. Yankelewitz (Eds.), *Children's reasoning while building fraction ideas* (pp. 83-94).

Van Ness, C. K., & Alston, A. S. (2017c). Switching the unit. In C. A. Maher & D. Yankelewitz (Eds.), *Children's reasoning while building fraction ideas* (pp. 65-81).

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). Academic.

Zhang, D., Stecker, P., & Beqiri, K. (2017). Strategies students with and without mathematics disabilities use when estimating fractions on number lines. *Learning Disability Quarterly*, *40*(4), 225-236. https://doi.org/10.1177/0731948717704966